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Interfaces with Other Disciplines

Choosing weights from alternative optimal solutions of dual multiplier models in DEA

William W. Cooper [a], José L. Ruiz [b,*], Inmaculada Sirvent [b]

[a] *The Red McCombs School of Business, The University of Texas at Austin, 1 University Station B6500, Austin, TX 78712-0212, USA*
[b] *Centro de Investigación Operativa, Universidad Miguel Hernández, Avd. de la Universidad, s/n, 03202-Elche (Alicante), Spain*

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Abstract

In this paper we propose a two-step procedure to be used for the selection of the weights that we obtain from the multiplier model in a DEA efficiency analysis. It is well known that optimal solutions of the envelopment formulation for extreme efficient units are often highly degenerate and, consequently, have alternate optima for the weights. Different optimal weights may then be obtained depending, for instance, on the software used. The idea behind the procedure we present is to explore the set of alternate optima in order to help make a choice of optimal weights. The selection of weights for a given extreme efficient point is connected with the dimension of the efficient facets of the frontier. Our approach makes it possible to select the weights associated with the facets of higher dimension that this unit generates and, in particular, it selects those weights associated with a full dimensional efficient facet (FDEF) if any. In this sense the weights provided by our procedure will have the maximum support from the production possibility set. We also look for weights that maximize the relative value of the inputs and outputs included in the efficiency analysis in a sense to be described in this article.

Keywords: Efficiency analysis; Data envelopment analysis (DEA); Efficiency evaluations; Weights and multipliers

1. Introduction

The dual (=“weight” or “multiplier”) models of DEA have not received extensive use like that made of the duality relations in other parts of linear programming where the optimal values of the dual variables have been widely used to examine the results of altering the model. This includes evaluating the desirability of altering the boundaries of constraining relations or determining rates of substitution and transformation. Important sources of trouble in the use of the dual models of DEA are represented by (1) the existence of alternate optima, which raises questions as to which multiplier values to use, and (2) the presence of numerous zeros

* Corresponding author. Tel.: +34 96 6658714; fax: +34 96 6658715.
E-mail address: jlruiz@umh.es (J.L. Ruiz).

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in some of these alternate optima that hamper usages for substitution and transformation analyses. See, for instance, Cooper et al. (2000a, 2001) and the cited references that justify turning to a use of non-zero slacks for purposes of substitution, and like, analyses.

A good description of these duality problems in DEA along with accompanying analytical characterizations may be found in Olesen and Petersen (1996, 2003) who we follow in their focus on identifying facets of the efficient frontier, in particular the full dimension efficiency facets (FDEFs) when they exist.[1] The facets of the efficient frontier are generated from the extreme efficient points, which are the ones designated as the set E in Charnes et al. (1991), and are fundamental in the sense that they are used to evaluate all of the points that represent the performances of the DMUs that are to be evaluated. In this paper, we supply models and methods for locating facets of the maximum possible dimension of the efficient frontier, in particular FDEFs when they are available, with the purpose of choosing weights as the coefficients of the associated supporting hyperplanes that contain the selected facets. Our method will also make it possible to determine a set of strictly positive multipliers, even when FDEFs do not exist.

Efficiency analyses are mainly concerned with obtaining the efficiency scores of a set of DMUs to be assessed and providing the corresponding peers and input and output deficiencies. However, analysts are sometimes interested in additionally estimating the weights with the purpose of obtaining added insight into the relative value of the inputs and outputs involved in synthesizing the overall "efficiency score" in such analyses.[2] A problem that often arises in practice, especially in the case of the extreme efficient units, is that we may have different optimal weights associated with the efficiency score of a given DMU and this may provide very different insights into the role played by the variables used in the efficiency assessment. In practice, the analysts generally use the weights provided by the software used, overlooking the possibility that there may be other optimal weights leading to very different conclusions. For this reason, in this paper we claim there is a need for selecting weights from among the optimal solutions of the dual multiplier models according to some criteria. In particular, we propose here a procedure, based on a couple of selection criteria, which can be used any time the value of the weights is of interest in an efficiency analysis.

This is organized as follows. In the next section, Section 2, we provide additional background on related developments such as the use of weight restrictions, as in Allen et al. (1997), and Thanassoulis and Allen (1998) or the "assurance region" approaches of Thompson et al. (1986) as well as the work by Bessent et al. (1988) and others concerned with the related problems of zero and non-zero slacks in alternate optima. In Section 3, we set forth the basic DEA models in both ratio and linear programming forms. In Section 4, we introduce a motivating example. In Section 5, we introduce model extensions for which we use mixed integer linear programming (MILP) formulations. This is followed in Section 6 by comparisons with other approaches. In Section 7, we introduce an empirical example which uses evaluations of the Rates Departments of the UK by Dyson and Thanassoulis (1988) and Thanassoulis et al. (1987) with results that we compare with our approach along with results obtained from different computer codes. We then draw conclusions that include comparisons with other approaches such as the one used in Charnes et al. (1991) which employs the strong complementary slackness condition (SCSC) of linear programming in order to secure non-zero weights. Finally, in Section 8, we summarize our results and suggest possible future directions of research that could involve joint uses of weight restrictions and the other approaches we describe.

2. Background

As previously noted, this paper is concerned with the selection of weights between the alternate optimal solutions of the dual multiplier formulation of the DEA models. When these alternative optimal solutions exist different optimal weights may be secured, depending on the software used.

Restricting weights is an approach commonly used in DEA to ensure that the resulting values are consistent with managerial or analyst views on the relative importance of inputs and outputs and/or to eliminate zeros.

[1] As Olesen and Petersen note such FDEFs may not exist, for instance because of the technologies used.

[2] Following Allen et al. (1997), here we focus on the weights as a "relative value system" of the variables involved in the analysis.

See, e.g., Allen et al. (1997) and Thanassoulis (2001) for purposes, motivations and uses of weight restrictions. However, the consequences of these choices (e.g., in terms of their effects on the efficiency scores) are generally not apparent a priori, and are not always easy to justify. As pointed out in Dyson and Thanassoulis (1988), it may be difficult to decide exactly how constraints on the weights are to be chosen for use in assessing various inputs and outputs.

We need to make clear at this point that the approach in this paper should not be viewed as an alternative that is inconsistent with imposing a priori weight restrictions. See, Cooper et al. (submitted for publication) for an example in which the two are used in complementary fashion. However, weight restrictions generally represent value judgements that are to be incorporated into the analysis on an a priori bases. This is in contrast to our procedure which uses a pair of general criteria of selection that is applicable to any data. Thus, one of the advantages of our approach is that it is a general method that can always be used. Finally, we make a choice of weights between different optimal solutions of the original DEA model and unlike a use of weight restrictions this does not alter the efficiency scores of the DMUs being evaluated.

In this paper, we restrict attention to extreme efficient points. As already noted, these points play a crucial role in DEA, since they are generally used to evaluate the performances associated with all of the other points. In fact, Cooper et al. (2000b, p. 209) prove that being DEA-efficient is a necessary condition for a DMU to participate actively (i.e., with a non-zero coefficient) as a peer in the evaluation of any DMU. In addition, it might be worth noting that inefficient DMUs rarely have alternative optimal weights in practice (i.e., the multiplier values for these DMUs usually have a unique optimal solution).

The procedure we propose is based on the geometrical idea of selecting weights associated with the efficient facets of the highest possible dimension of the frontier that the DMU under assessment contributes to span. This means that the weights will be associated with FDEFs of the frontier, when they exist. In that sense, the criterion of selecting weights by maximizing the dimension of the associated facets of the efficient frontier will generate optimal weights having the maximum possible support from the data, even when the obtained facets are not of full dimension. This might equivalently be stated as a selection of weights associated with hyperplanes that maximize contact with the production possibility set.

In a second stage, undertaken after the previously selected optimum solutions are obtained, we look for those weights that represent performance evaluations in which all variables considered in the model simultaneously maximize their relative values. To be more specific, we maximize the relative value of the variable with minimum value. In the absence of any preferences on the importance of the variables used, this criterion leads to the most well-balanced set of virtual inputs and outputs of the assessed unit.[3] It is to be noted that, as a result, we obtain an optimal solution with no zero weights so that no variable is completely ignored in the relative efficiency assessment. Thus, our approach can also be of help in dealing with zero weights.

The approach in Charnes et al. (1991) is related to the one in the present paper. The authors of that paper select optimal weights satisfying the strong complementary slackness condition (SCSC), which is especially well adapted for extreme efficient units (as in our case). This ensures strictly positive weights but, in contrast to our approach, the hyperplane associated with the resulting weights is supported only by the assessed unit and, as will be shown later in this paper, this means that it has minimum support from the observed data.

Other related papers include those by Bessent et al. (1988), with an approach they call "constrained facet analysis". That paper and the paper by Chang and Guh (1991) deal with the problem of accounting for slacks in the efficiency scores. Green et al. (1996) identify some problems with these approaches and propose an alternative that is aimed at providing efficiency bounds with reference to FDEFs of the frontier (when they exist) that is closely related to the work in Olesen and Petersen (1996). Like Chang and Guh (1991), Chen et al. (2003) also use SCSC in order to replace the non-Archimidean infinitesimal ε by a specific finite magnitude that makes it possible to assess these units with reference to a set of strictly positive weights. Unlike our approach as well as the approach by Charnes et al. (1991), these papers focus mainly on the assessment of inefficient units having non-zero slacks in such alternate optima.

[3] This criterion has already been used in the literature (see, Thanassoulis, 2001).

3. DEA models

We now turn to our proposed approach. Suppose we have a set of n DMUs each of which utilizes m inputs to produce s outputs. The inputs and outputs for all of the DMUs are assumed to be strictly positive when the relative efficiency of each DMU_0, is assessed by using the following DEA model (Charnes et al., 1978):

$$\begin{aligned}
\text{Max} \quad & \frac{\sum_{r=1}^{s} u_r y_{r0}}{\sum_{i=1}^{m} v_i x_{i0}} \\
\text{s.t.} \quad & \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \leqslant 1, \quad j = 1, \ldots, n, \\
& v_i \geqslant 0, \quad i = 1, \ldots, m, \\
& u_r \geqslant 0, \quad r = 1, \ldots, s,
\end{aligned} \tag{1}$$

where the y_{r0}, $r = 1, \ldots, s$ represent outputs and the x_{i0}, $i = 1, \ldots, m$ represent inputs for each of $j = 1, \ldots, n$, DMUs and $j = 0$ identifies the DMU_j to be evaluated.

Since (1) is a linear fractional programming problem (see Charnes and Cooper, 1962) we can transform it into an ordinary linear programming problem by proceeding as follows: First we note that multiplying both numerators and denominators in the objective and in the constraints by $\beta > 0$ leaves their values unchanged. For this purpose we use $\beta = (\sum_{i=1}^{m} v_i x_{i0})^{-1}$. We then define the following new variables $\mu_r = \beta u_r$, $\nu_i = \beta v_i$ and obtain the following ordinary linear programming problem which has the same optimum value as (1)[4]

$$\begin{aligned}
\text{Max} \quad & \omega_0 = \sum_{r=1}^{s} \mu_r y_{r0} \\
\text{s.t.} \quad & \sum_{i=1}^{m} \nu_i x_{i0} = 1, \\
& -\sum_{i=1}^{m} \nu_i x_{ij} + \sum_{r=1}^{s} \mu_r y_{rj} \leqslant 0, \quad j = 1, \ldots, n, \\
& \nu_i \geqslant 0, \quad i = 1, \ldots, m, \\
& \mu_r \geqslant 0, \quad r = 1, \ldots, s.
\end{aligned} \tag{2}$$

This problem has a dual which takes the form of an "envelopment model" that can be written

$$\begin{aligned}
\text{Min} \quad & \theta_0 \\
\text{s.t.} \quad & \sum_{j=1}^{n} x_{ij} \lambda_i \leqslant x_{i0} \theta_0, \quad i = 1, \ldots, m, \\
& \sum_{j=1}^{n} y_{rj} \lambda_j \geqslant y_{r0}, \quad r = 1, \ldots, s, \\
& \lambda_j \geqslant 0, \quad j = 1, \ldots, n.
\end{aligned} \tag{3}$$

This problem always has a solution by choosing θ_0, $\lambda_j = 1$ for the $\mathrm{DMU}_j = \mathrm{DMU}_0$ that is to be evaluated and all other $\lambda_j = 0$. Moreover, the solution is bounded below by $\max y_{r0}$, $r = 1, \ldots, s$, and the $\lambda_j \geqslant 0$ constraints. Hence the optimal value of θ_0 is finite so that, via the dual theorem of linear programming, model (2) has a finite solution with $\omega_0^* = \theta_0^*$ where "$*$" indicates an optimal value.

Model (3) is the "Farrell model", due to Farrell (1957), because it does not include slacks in the objective, as is done in the CCR model formulated by Charnes et al. (1978). We omit this feature of the CCR model because we are directing attention to points $E_j \in E$ and such points have a solution with zero slacks. In fact

[4] We are here using the "input oriented" version but note that by reciprocating the numerator and denominator of (1) we could equally well have used the "output oriented" version in the analyses that follow.

with the commonly used extreme point methods of solution (such as the simplex method) this solution will take the form $\lambda_j^* = 1$ and all other variables will have a value of zero.

4. Motivation: Empirical example

To help supply insight into the problems we are dealing with, we turn to a concrete example for which we use the data that Thanassoulis et al. (1987) employed to assess the Metropolitan and London Borough Rates departments. Since they were interested in all observations, efficient or not, Thanassoulis et al. used the CCR version, as in (3), to secure their performance evaluations. These evaluations were revisited in Dyson and Thanassoulis (1988) for the purpose of illustrating the use of weight restrictions to reduce the total weight flexibility in DEA and, in particular, to avoid zero weights.

The sample used in both of the above cited studies consists of 62 departments treated as DMUs and assessed with reference to one input (total cost of rates collection) and four outputs (non-council hereditaments, rates rebates granted, summonses issued and distress warrants obtained, and net present value (NVP) of non-council rates collected). The DEA analysis in Dyson and Thanassoulis (1988) identified seven departments as extreme efficient units: Lewisham, Brent, Stockport, Bradford, Leeds, City of London and Liverpool.

To avoid clutter we do not reproduce the results for all 62 departments. Instead we focus on only these seven efficient departments. Table 1 therefore records the optimal weights reported in the Dyson and Thanassoulis paper – see row (D&T) for each of these efficient department. It also provides the weights obtained from these same data by two different DEA-specific softwares (the Efficiency Measurement Software

Table 1
Optimal weights from different software

Department	Software	Non-cln hereds	Rates rebat. granted	Summ. & dist. wrnts.	NPV of rts collected
Lewish.	D&T	0	0.0854	0.2640	0.1095
	EMS	0.0215	0.0883	0.2713	0
	DEA-Solver	0.0212	0.0661	0.3049	0.0058
	LINDO	0	0.0973	0.2646	0
Brent	D&T	0	0.0369	0.3222	0.1338
	EMS	0	0	0.3634	0.0616
	DEA-Solver	0.0240	0.0293	0.3521	0.0066
	LINDO	0	0	0.3781	0
St'port	D&T	0.2960	0.0559	0.1547	0
	EMS	0.5280	0	0	0
	DEA-Solver	0.3172	0.0516	0.1200	0.0460
	LINDO	0.5280	0	0	0
B'ford	D&T	0	0.0830	0.2598	0.1357
	EMS	0.1055	0.0449	0.2443	0.1160
	DEA-Solver	0.2827	0.0518	0.1523	0.0462
	LINDO	0.2674	0	0.2467	0
Leeds	D&T	0	0.0830	0.2598	0.1357
	EMS	0.2397	0.0758	0.0632	0.1099
	DEA-Solver	0.3172	0.0516	0.1200	0.0460
	LINDO	0.2944	0.0532	0.1593	0
City of London	D&T	0	0.0830	0.2598	0.1356
	EMS	0	0	0	0.1448
	DEA-Solver	0.2891	0.0492	0.1144	0.1276
	LINDO	0.4723	0	0	0.1233
Liver'l	D&T	0	0.0809	0.2725	0.0993
	EMS	0	0.0413	0.3428	0.0075
	DEA-Solver	0.0212	0.0661	0.3049	0.0058
	LINDO	0.0270	0.0287	0.3533	0

(EMS) and the DEA-Solver Pro)[5] and, finally, the weights obtained from (LINDO) a conventional mathematical programming software. From the results shown in this table it is apparent that the optimal solutions yield weights that are very different depending on the software used. We should therefore be cautious when interpreting the thus obtained weights, since different software may select different optimal values from the different alternate optima and lead to different conclusions on the relative value of the inputs and outputs involved. Especially to be noticed is that for some departments, such as Brent and City of London, we have optimal solutions with three, two, one and even with no zero weights.

We might note that optimal solutions with no zero weight are always achieved by DEA-Solver but not by EMS or LINDO. Using model (2), Thanassoulis et al. (1987) always exhibit one non-zero weight with their use of Fortran programs as reported in the rows labelled D&T in Table 1. (See p. 400 in Thanassoulis et al. (1987) for a description of their algorithmic choices.)

This leads us to methods for reducing flexibility in the choices of weights – in particular because these zeros cause "one to effectively ignore some inputs or outputs", as Dyson and Thanassoulis note. For this purpose we turn in a different direction and utilize extensions and modifications of (2) in order to provide optimal solutions for the multipliers with non-zero weights. We confine attention to choices between alternate optima so that the original efficiency scores are not affected by the solution we obtain. Hence, unlike what can occur in a use of weight restrictions, the efficient points E that we use will remain efficient and can therefore continue to be used to evaluate the other DMUs.

5. The two-step procedure

As said before we are confining attention to the extreme efficient units. These are the points that are designated as E, in contrast to E', F, and NF in Charnes et al. (1991, p. 201). The existence of alternate optimal weights for extreme efficient units suggests the need for establishing a criterion of general applicability that can be used for choosing weights by selections from the optimal solutions for the multipliers in (2).

It should be noted that we might also have multiple optimal weights for efficient units that are not extreme (i.e., points in E'), if they are on a facet that is not of full dimension. Consequently, the approach in this paper could also be used for these DMUs – although this type of DMU is rarely found in practice. Moreover, our approach can also be adapted for use with inefficient DMUs – although these DMUs usually have unique solutions for the weights when solving (2).

To describe our approach we proceed in the following two-step manner:

- *First step:* From the optimal solutions to (2) we select those associated with the hyperplanes that are supported by the maximum number of extreme efficient units. This means that we select weights that are associated with the facets of the frontier of highest dimension that the unit under assessment contributes to span. In this sense, the weights obtained from these solutions in this manner will have the maximum possible support from the data, as was discussed in Section 2, above.
- *Second step:* From the weights chosen in the first step, we select those which maximize the minimum relative value of the variables involved as measured by the corresponding "virtual variable". In this step, we look for weights that have associated programs of performance which, in the terminology of Dyson and Thanassoulis, maximize their "importance" relative to all of the variables taken together (see also Thanassoulis, 2001). For efficient units, it will be shown that it is always possible to find a set of weights with the maximum support from the data in which no variable is completely ignored.

For any extreme efficient DMU_0 the two-step procedure above is implemented as follows:

- *First step:* The following mixed integer linear programming (MILP) problem selects between the optimal weights obtained for DMU_0, when solving (2), those associated with the hyperplanes that are supported by the maximum possible number of extreme efficient units

[5] EMS uses an interior point solver called BPMPD (a primal–dual interior point algorithm) while DEA-Solver Pro uses the Solver tool of Excel.

$$
\begin{aligned}
\text{Min}\quad & I_0=\sum_{j\in E} I_j\\
\text{s.t.}\quad & \sum_{i=1}^{m} v_i x_{i0}=1,\\
& -\sum_{i=1}^{m} v_i x_{ij}+\sum_{r=1}^{s}\mu_r y_{rj}+t_j=0,\quad j\in E,\\
& \sum_{r=1}^{s}\mu_r y_{r0}=1,\\
& t_j-MI_j\leqslant 0,\quad j\in E,\\
& I_j\in\{0,1\},\quad j\in E,\\
& v_i\geqslant 0,\quad i=1,\ldots,m,\\
& \mu_r\geqslant 0,\quad r=1,\ldots,s,\\
& t_j\geqslant 0,\quad j\in E,
\end{aligned}
\tag{4}
$$

where M is a big positive quantity – e.g., it suffices to choose $M=\sum_{i=1}^{m}\frac{\max_{j\in E}\{x_{ij}\}}{\min_{j\in E}\{x_{ij}\}}$.

The guiding idea of (4) is the following: Firstly, via the first three groups of constraints we consider all the optimal solutions of (2); secondly, note that $I_j=0$ implies $t_j=0$, i.e., DMU$_j$ belongs to the hyperplane $-\sum_{i=1}^{m}v_ix_i+\sum_{r=1}^{s}\mu_ry_r=0$. Then, since we are minimizing $\sum_{j\in E}I_j$ and $I_j\in\{0,1\}$, (4) will be directed to finding optimal solutions with as many $I_j^*=0$ as possible, i.e., with as many $t_j^*=0$ as possible. In other words, in solving (4) we are choosing those solutions that have the maximum number of $t_j^*=0$, i.e., those hyperplanes at the point associated with DMU$_0$ that are supported by the maximum number of extreme efficient DMUs.

This is depicted in Fig. 1. This figure can be viewed as representing a section at a given output level, say $y=1$, of the production possibility set generated by three DMUs (A, B and C) that use two inputs and produce the same quantity of output ($y=1$). The optimal solutions of (2) when assessing the efficiency of the extreme efficient DMU B correspond to the coefficients of the supporting hyperplanes at B, which pass through the origin, that form a family which starts at the one that contains the solid line connecting A to B and continues through rotations about B (represented with dotted lines) until coincidence is achieved with the hyperplane containing the solid line connecting B to C. Model (4) then selects one of the two hyperplanes represented with a dark solid line as distinct from the ones represented by the lighter dotted lines. The first two are obviously preferable to the latter because they are supported by two units (either A and B or B and C)

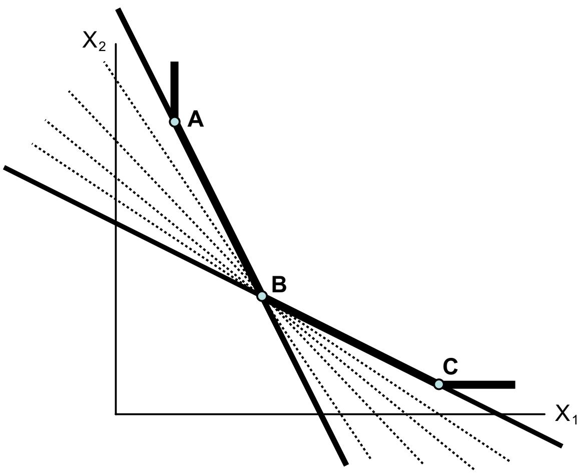

Fig. 1. Supporting hyperplanes and efficiency frontiers.

instead of by only one (B). Moreover, in this particular case, this also means that they contain a FDEF of the frontier that DMU B contributes to generate. If any of the other optimal weights were chosen then we would be selecting a hyperplane that intersects the production possibility set only at the ray from the origin through B, whereas the two solutions provided by (4) have in common with the production possibility set a facet (of maximal dimension) of its frontier. In this sense, these two choices can be regarded as maintaining maximum contact with the frontier of the production possibility set from which evaluations are being made.

From a geometrical point of view, for each extreme efficient unit DMU_0, model (4) selects the weights that are associated with hyperplanes containing the facets of higher dimension of the efficient frontier to which DMU_0 belongs.[6] In particular, if the optimal value of I_0^* in (4) equals $|E| - (m+s-1)$ – where $|E|$ represents the cardinality of E – then an optimal solution of (4) will be associated with a hyperplane containing a FDEF which DMU_0 contributes to span. Thus, we can see that if DMU_0 is located on some FDEF then (4) will yield a solution that corresponds to it. This will be the best solution for the weights since they will then have the maximum possible support from the data that can be achieved for the given dimensionality of the input–output space. Even in this case, (4) will yield alternate optima if DMU_0 contributes to span more than one FDEF. Nevertheless, in this situation we will have only a finite number of alternative optimal weights, since there is a unique supporting hyperplane containing each FDEF.[7]

If, on the contrary, $I_0^* > |E| - (m+s-1)$, then DMU_0 is not located on any FDEF of the efficient frontier. Therefore, (4) will provide weights associated with hyperplanes containing the facets of higher possible dimension (with the current frontier) that this unit generates, and this will be strictly lower than $m+s-1$. In this case, we will have an infinite choice of possible optimal weights for DMU_0 associated with each of these facets.

Remark. It should also be noted that (4) is similar to the model that is used as a general test for the existence of FDEFs on the efficient frontier in Olesen and Petersen (1996). The difference is that we solve (4) for each extreme efficient unit with the purpose of selecting hyperplanes containing the facets of the higher possible dimensions of the efficient frontier that this unit contributes to span, which are not necessarily of full dimension.

The possibility of alternate optimal solutions in (4) brings us to the second step in our procedure. In this step we refine the selection of optimal weights made in the first step by choosing those that satisfy an extra criterion.

• *Second step:* The following MILP problem selects between the alternative optima (if any) provided by (4), by maximizing the minimum value of the virtual variables.

$$
\begin{aligned}
\text{Max} \quad & z_0 \\
\text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{i0} = 1, \\
& -\sum_{i=1}^{m} v_i x_{ij} + \sum_{r=1}^{s} \mu_r y_{rj} + t_j = 0, \quad j \in E, \\
& \sum_{r=1}^{s} \mu_r y_{r0} = 1, \\
& t_j - M I_j \leqslant 0, \quad j \in E, \\
& \sum_{j \in E} I_j = I_0^*,
\end{aligned}
$$

[6] In order to use (4) for determining the dimension of the facets associated with its optimal solutions for the multipliers we have to assume that each subset of $m+s-1$ DMU_j's in E is a set of $m+s-1$ linear independent $(m+s)$-vectors (or all of the DMUs in E if this set has less than $m+s-1$ units).

[7] Note that in the case of extreme efficient units located on a FDEF, the set of feasible weights of (4) will coincide with those for the approach in Green et al. (1996).

$$
\begin{aligned}
& v_i x_{i0} \geqslant z_0, \quad i = 1, \ldots, m, \\
& \mu_r y_{r0} \geqslant z_0, \quad r = 1, \ldots, s, \\
& I_j \in \{0, 1\}, \quad t_j \geqslant 0, \; j \in E, \\
& z_0 \geqslant 0, \\
& v_i \geqslant 0, \quad i = 1, \ldots, m, \\
& \mu_r \geqslant 0, \quad r = 1, \ldots, s,
\end{aligned} \tag{5}
$$

where I_0^* is the optimal value of (4).[8]

Model (5) is constrained to maintain the optimal $\min I = I_0^*$ obtained from (4). The constraints that were satisfied in (4) are also maintained in (5). Hence the solution to (5), including possible alternate optima, are also maintained from (4).

Among these solutions we now seek to maximize the minimum virtual variable and this is accomplished by the new variable z_0 that is to be maximized. As can be seen, this objective is directed to maximizing the minimum value of these $m + s$ new constraints that represent all of these ''multipliers'' (or ''weights'') under the condition that the solutions must also satisfy the constraints of (4). Thus, the weights obtained represent performance evaluations in which all variables considered in the model simultaneously maximize their relative values.[9] Moreover, with these weights the variables will be maximizing their contribution to the efficiency score of the assessed unit.

Now, by virtue of the following theorem we will have a solution in which all of these weight variables will have positive values:

Theorem 1. *For any efficient DMU the optimal value of (5) is strictly greater than zero, i.e.,* $z_0^* > 0$.

Proof. See Appendix.

6. Comparisons with other approaches

We now relate our approach to preceding work by others. By definition, see e.g., Olesen and Petersen (1996), for any efficient unit we can always find an optimal solution with all weights strictly positive. Our theorem carries this further by saying that this is also possible for weights that have maximum support from the data and, furthermore, we also show how this may be accomplished.

Turning to the selection of optimum weights, Charnes et al. (1991) provide a solution method involving SCSC[10] that generates strictly positive optimal weights. However, in contrast to our approach, their use of the SCSC condition yields solutions that are associated with hyperplanes that have minimal support from the data. This can be seen as follows: The SCSC states that $\lambda_j^* + t_j^* > 0$, and $\lambda_j^* t_j^* = 0$, $j = 1, \ldots, n$, where λ_j^* is an optimal solution for (3) and t_j^* is the optimal slack for the constraint $-\sum_{i=1}^m v_i^* x_{ij} + \sum_{r=1}^s \mu_r^* y_{rj} \leqslant 0$ in (2). Since DMU_0 is an extreme efficient unit, the optimal solutions of (3) satisfy $\lambda_0^* = 1$ and $\lambda_j^* = 0$, for any DMU_j different from DMU_0. By SCSC, this implies that $t_j^* > 0$ for any DMU_j different from DMU_0, i.e., that the associated supporting hyperplane intersects the observations only at DMU_0. In contrast to our solutions, which have as many $t_j^* = 0$ as possible, such solutions have only minimal contact with the data.

Finally, we note the procedure in Thanassoulis (2001) already mentioned in this paper, is directed to provide weights regarding what he calls ''robustness'' of the efficiency rating, which can be judged by the extent to which the assessed unit relies on a limited number of inputs and outputs to be rated Pareto-efficient. Unlike in (5), this procedure takes into account all the alternate optima for the weights and not only those maximizing the dimension of the associated facet of the frontier (i.e., only the optimal solutions of (4)).

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[8] Formulations (4) and (5) can be easily solved by using conventional mathematical programming software such as LINDO/LINGO or the Solver tool of Excel.

[9] It should be noted that maximizing the virtuals, instead of the absolute weights, ensures that the resulting virtuals of inputs and outputs will be invariant to the units of measurement.

[10] The SCSC is also referred to as the ''Extended Theorem of the Alternative''. See p. 441 in Charnes and Cooper (1961).

7. Empirical example

To illustrate the use of the methodology we have presented in this paper we revisit the data in Thanassoulis et al. (1987) that we have already described.

In Table 2 we have recorded the optimal weights reported in Dyson and Thanassoulis (1988). These values are recorded for each Department on line 1 as D&T. The rows labelled (4) and (5) record solutions corresponding to each of the two models that we have just developed in our two-step procedure. In the last column of the table we also record the extreme efficient DMUs that support the efficient facet corresponding to these optimal solutions. As can be seen, the solutions associated with (5) yield positive values for all variables.

Table 3 shows the values $\mu_r^* y_{r0}$ of the virtual outputs associated with the optimal weights both for Dyson and Thanassoulis and those obtained in the second step in our procedure (i.e., model (5)). These values provide us with insight into the contribution of each variable to the efficiency score of the assessed department. As expected, the virtual variables in the second step using (5) are all strictly positive. In particular, the virtuals obtained from model (5) for Stockport, Bradford, Leeds and, perhaps, City of London show that these four departments can accommodate the weights in such a way that the variables involved play a role in the overall efficiency assessment.

In all cases, some of the outputs that had a zero weight in the conventional DEA assessment reported in the Dyson and Thanassoulis paper now make a positive contribution to the efficiency evaluations. See for example, the NPV of rates collected in Stockport, which has an associated virtual value of 0.1096, and that of non-council hereditaments in Bradford, Leeds and City of London, whose virtuals are 0.3871, 0.3835 and 0.0823 respectively. Turning to Lewisham, Brent and Liverpool, we note that some of these virtual output values are so low that they might be excluded without much effect on their scores. We could even infer that these DMUs might have been directing their efforts to the activity of issuing summons and warrants where, in all the cases, the virtual value of this activity is relatively high. In contrast, when weight restrictions are imposed on other activities, as in Dyson and Thanassoulis (1988), these three departments become inefficient. That is their efficiency score is changed.

Table 2
Optimal weights in Dyson and Thanassoulis (1988) and the two-step procedure

Department	Model	Non-cln hereds	Rates rebat. granted	Summ. & dist. wrnts.	NPV of rts collected	Depts. supporting
(l) Lewisham	D&T	0	0.0854	0.2640	0.1095	
	(4)	0.0423	0.0868	0.2664	0	1, 4, 5
	(5)	0.0188	0.0876	0.2668	0.0369	1, 4, 5
(2) Brent	D&T	0	0.0369	0.3222	0.1338	
	(4)	0	0.0228	0.3313	0.1336	2, 4, 6
	(5)	0.0256	0.0091	0.3345	0.1336	2, 4, 6
(3) St'port	D&T	0.2960	0.0559	0.1547	0	
	(4)	0.2618	0.0494	0.1398	0.1280	3, 4, 5, 6
	(5)	0.2618	0.0494	0.1398	0.1280	3, 4, 5, 6
(4) B'ford	D&T	0	0.0830	0.2598	0.1357	
	(4)	0.2618	0.0494	0.1398	0.1280	3, 4, 5, 6
	(5)	0.2618	0.0494	0.1398	0.1280	3, 4, 5, 6
(5) Leeds	D&T	0	0.0830	0.2598	0.1357	
	(3)	0.2618	0.0494	0.1398	0.1280	3, 4, 5, 6
	(4)	0.2618	0.0494	0.1398	0.1280	3, 4, 5, 6
(6) City of London	D&T	0	0.0830	0.2598	0.1356	
	(4)	0.2618	0.0494	0.1398	0.1280	3, 4, 5, 6
	(5)	0.2618	0.0494	0.1398	0.1280	3, 4, 5, 6
(7) Liver'l	D&T	0	0.0809	0.2725	0.0993	
	(4)	0.0270	0.0287	0.3534	0	2, 4, 7
	(5)	0.0165	0.0309	0.3489	0.0230	2, 4, 7

Table 3
Virtual outputs

Department	Model	Non-cln hereds	Rates rebat. granted	Summ. & dist. wrnts.	NPV of rts collected
(l) Lewisham	D&T	0	0.3191	0.6349	0.0461
	(5)	0.0155	0.3273	0.6417	0.0155
(2) Brent	D&T	0	0.0631	0.8521	0.0849
	(5)	0.0156	0.0156	0.8847	0.0840
(3) St'port	D&T	0.5606	0.1300	0.3096	0
	(5)	0.4958	0.1147	0.2798	0.1096
(4) B'ford	D&T	0	0.2719	0.6368	0.1150
	(5)	0.3871	0.1617	0.3428	0.1084
(5) Leeds	D&T	0	0.5106	0.3940	0.1069
	(5)	0.3835	0.3036	0.2120	0.1008
(6) City of London	D&T	0	0.0023	0.0604	0.9363
	(5)	0.0823	0.0014	0.0325	0.8838
(7) Liver'l	D&T	0	0.2658	0.6840	0.0497
	(5)	0.0115	0.1014	0.8756	0.0115

$\mu_r^* y_{r0}$ with $\sum_{r=1}^{s} \mu_r^* y_{r0} = 1$ for (5).

Returning to Table 2 we see that Stockport, Bradford, Leeds and the City of London are all evaluated by the same optimal weights. To be specific, these weights are associated with a hyperplane that contains a facet of the efficient frontier on which these four departments are located. Moreover, with four outputs and one input we have $(m + s - 1) = 1 + 4 - 1 = 4$ showing that these departments are all associated with a FDEF of the frontier that is spanned by these four departments (and the origin) and this is the case for both procedures (4) and (5).

This is not the case for the other three departments: Lewisham, Brent and Liverpool. The performances of these DMUs are all evaluated by weights that are associated with hyperplanes supported by only three DMUs. Hence they are not on a FDEF and, despite this deficiency, the procedure associated with (5) yields non-zero weights.

As previously discussed, with the current frontier, the weights provided by our procedure have the maximum support from the observations. We have seen that, in the particular case of Stockport, Bradford, Leeds and City of London, that is the maximum possible support for the dimension of the input–output space, since the optimal weights are associated with a supporting hyperplane containing a FDEF. We note that this does not happen with the optimal weights provided in the Dyson and Thanassoulis paper for these four departments, because they are different from the ones provided by our procedure and there is a unique FDEF in the efficient frontier.

Dyson and Thanassoulis (1988) considered the original optimal weights to be unacceptable because they had many unrealistic zeros. They therefore decided to impose lower bounds on some of these weights. The fact that only one input was involved made it possible to use a regression analysis for their purposes. (See the Dyson and Thanassoulis paper for details.) The lower bounds that were used are recorded at the bottom of Table 4. This table also records the optimal weights resulting from the restricted DEA model provided in that paper and, again, the weights from the second step of our procedure as taken from Table 2.

First we note that Lewisham, Brent and Liverpool all became inefficient, as indicated by the * alongside the w.r. (=weight restriction) used, which means that this change of status occurred when the weight restrictions were imposed. This did not occur for Stockport, Bradford, Leeds or the City of London where the weights all became positive, as desired by D&T.

Turning to the values from (5), as transferred to Table 4 from Table 2, we see that these values are all fairly close to those reported by D&T for the efficient but not for the inefficient performers. For the former, our

Table 4
Optimal weights for (5) and for (D&T) with weight restrictions

Department	Model	Non-cln hereds	Rates rebat. granted	Summ. & dist. wrnts.	NPV of rts collected
(1) Lewisham	D&T (w.r.)*	0.2520	0.0531	0.1580	0.0970
	(5)	0.0188	0.0876	0.2668	0.0369
(2) Brent	D&T (w.r.)*	0.2520	0.0392	0.1740	0.0970
	(5)	0.0256	0.0091	0.3345	0.1324
(3) St'port	D&T (w.r.)	0.2520	0.0531	0.1580	0.0970
	(5)	0.2618	0.0494	0.1398	0.1280
(4) B'ford	D&T (w.r.)	0.2520	0.0532	0.1570	0.0970
	(5)	0.2618	0.0494	0.1398	0.1280
(5) Leeds	D&T (w.r.)	0.2520	0.0531	0.1580	0.0970
	(5)	0.2618	0.0494	0.1398	0.1280
(6) City of London	D&T (w.r.)	0.2520	0.0521	0.1460	0.1282
	(5)	0.2618	0.0494	0.1398	0.1280
(7) Liver'l	D&T (w.r.)*	0.2520	0.0531	0.1580	0.0970
	(5)	0.0165	0.0309	0.3489	0.0230
Lower bounds for the weights		0.252	0.0392	0.088	0.097

* The department becomes inefficient with w.r.

approach shows that there was a suitable choice of optimal weights for these departments among the original optimal solutions of (2). On one hand, our weights satisfy the lower bounds imposed and, on the other, our weights are associated with a FDEF. For the latter, D&T report the following new values in place of their former (unity value) efficiency scores:

Department	Efficiency
Lewisham	0.827
Brent	0.743
Liverpool	0.796

Thus the reductions in efficiency range from slightly in excess of 17% in the case of Lewisham to nearly 26% in the case of Brent.

As D&T (1988, p. 570) note, the values to be used for such restrictions "are difficult to determine in practice" and the resulting reductions in efficiency may be difficult to justify when challenged. Use of our two-step procedure produces non-zero weights that can be used both to guide the choice of weight restrictions and also to provide additional justification for the resulting choices in that they are (a) dependent only on the data and (b) reflect a use of the data in all of the relevant dimensions for the activities being evaluated.

Remark. Cooper et al. (submitted for publication) report an application to evaluating performances of the players on a basketball team in the Spanish basketball league in which the weights derived by our procedures were found to be entirely consistent with the weights supplied by the coach in accordance with the Thompson et al. (1986) assurance region approach. Of special interest to the coach and technical staff was the ability of our approach to identify good "all around" players with relatively good scores in all attributes and distinguish them from players with the same efficiency score but, in the latter case, this involved weights with very high values in one or more attributes, such as "steals" or "free throws" and this was accompanied by low values in other attributes. The coach and technical staff found this much more informative than what was obtained in the form of simply an overall evaluation such as they customarily obtained by using the official index used by the league.

Finally, we should mention that we believe that model (5) will rarely have alternative optimal solutions in practice, since the two implemented selection criteria drastically reduce the choices in the weights, although this uniqueness cannot be guaranteed theoretically. In fact, we have theoretically checked that (5) has unique optimal solutions for the case of the data used in this empirical illustration, which are those in the row labelled (5) in Tables 2 and 4.

8. Conclusions

In this paper, we have proposed a two-step procedure that is intended to yield a choice of weights from among the available possibilities offered by the alternate optima in the dual "multiplier" formulation of a DEA model according to a pair of criteria. Traditionally, weights are used and immediately interpreted as they are provided by the software used, for instance, without considering that there may be alternative optimal weights leading to very different conclusions. Our empirical illustration has shown that the set of alternate optima for the weights deserves to be explored before any conclusions are drawn and that our procedure can help in making a choice of weights. Our selection is further justified because of the support that the estimated weights receive from the observed data. To accomplish this, we look for weights associated with hyperplanes supported by the maximum number of extreme efficient units. This selection is complemented in a second step in which we choose weights that "maximize" the relative value of the inputs and outputs selected for the efficiency assessment. In particular, our theorem guarantees that no variable will be completely ignored in the assessment of efficient units. Obviously, other criteria might be used to select weights, and so, our work should be viewed as contributing to progress in this context.

The literature dealing with the selection of optimal weights is sparse. The approach in Charnes et al. (1991) is closely related to the approach presented here. The authors of that paper develop a procedure that, by virtue of SCSC, guarantees to provide strictly positive optimal weights. However, the SCSC condition leads to hyperplanes that have minimum support from the data, since they are supported only by the assessed unit. Thanassoulis (2001) makes a choice of optimal weights with respect to the robustness of the efficiency scores. In other related papers, we can find several approaches that implicitly make a selection of weights, but with the aim of measuring the efficiency, especially that of inefficient DMUs with non-zero slacks.

A suitable choice of weights may be of interest in other related problems. For instance, Allen et al. (1997, p. 21) point out that authors generally do not clarify how to treat alternative optima with the purpose of specifying bounds for weights restrictions. Our approach contributes to that objective in its considerations of alternate optima (see also Olesen and Petersen, 2003). We also mention the possibility of extending our approach for use with models with restrictions on the weights. Thus, our procedure can contribute to making a choice from among the optimal weights satisfying the conditions resulting of incorporating value judgements into the models without affecting the overall efficiency score. Other extensions could include adjunction of the convexity condition $\sum_{j=1}^{n}\lambda_j = 1$ and the use of non-discretionary variables in model (3). These and other possibilities represent future directions for research.

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Appendix. Proof of the theorem

We first prove the theorem for any extreme efficient DMU_0 and then extend the proof to any non-extreme efficient DMU_0.

For the sake of simplicity, let us assume that the first e units of the set of n DMUs are the extreme efficient ones. Let e–k be the optimal value of (4) associated with the extreme efficient unit DMU_0. Thus, we can also assume (for simplicity) that $I_1^* = \cdots = I_k^* = 0$ and $I_{k+1}^* = \cdots = I_e^* = 1$ are the values of the binary variables in (4) associated with a given optimal solution. Then, to prove $z_0^* > 0$, it suffices to show that the optimal value of the following LP problem is greater than zero:

$$\begin{aligned}
\text{Max} \quad & z_0 \\
\text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{i0} = 1, \\
& -\sum_{i=1}^{m} v_i x_{ij} + \sum_{r=1}^{s} \mu_r y_{rj} = 0, \quad j = 1, \ldots, k, \\
& -\sum_{i=1}^{m} v_i x_{ij} + \sum_{r=1}^{s} \mu_r y_{rj} \leqslant 0, \quad j = k+1, \ldots, e, \\
& v_i \geqslant z_0, \quad i = 1, \ldots, m, \\
& \mu_r \geqslant z_0, \quad r = 1, \ldots, s, \\
& z_0 \geqslant 0, \\
& v_i \geqslant 0, \quad i = 1, \ldots, m, \\
& \mu_r \geqslant 0, \quad r = 1, \ldots, s,
\end{aligned} \tag{A.1}$$

since for a given optimal solution $(v_1^*, \ldots, v_m^*, \mu_1^*, \ldots, \mu_s^*, z_0^*)$ of (A.1) satisfying $z_0^* > 0$ we can always construct the following feasible solution of (5) which has an associated objective value that is strictly positive: $v_i = v_i^*$, $i = 1, \ldots, m$, $\mu_r = \mu_r^*$, $r = 1, \ldots, s$, $I_j = 0$, $j = 1, \ldots, k$, and $I_j = 1$, $j = k+1, \ldots, e$, $t_j = 0$, $j = 1, \ldots, k$, $t_j = \sum_{i=1}^{m} v_i^* x_{ij} - \sum_{r=1}^{s} \mu_r^* y_{rj}$, $j = k+1, \ldots, e$, and $z_0 = z_0^* \cdot \min_{\substack{i=1,\ldots,m \\ r=1,\ldots,s}} \{x_{i0}, y_{r0}\}$.

The dual formulation of (A.1) is

$$\begin{aligned}
\min \quad & \theta_0 \\
\text{s.t.} \quad & \sum_{j=1}^{k} \lambda_j x_{ij} + \sum_{j=k+1}^{e} \alpha_j x_{ij} \leqslant \theta_0 x_{i0} - s_{i0}^-, \quad i = 1, \ldots, m, \\
& \sum_{j=1}^{k} \lambda_j y_{rj} + \sum_{j=k+1}^{e} \alpha_j y_{rj} \geqslant s_{r0}^+, \quad r = 1, \ldots, s, \\
& \sum_{i=1}^{m} s_{i0}^- + \sum_{r=1}^{s} s_{r0}^+ \geqslant 1, \\
& \theta_0 \quad \text{free}, \\
& \lambda_j \quad \text{free}, \quad j = 1, \ldots, k, \\
& \alpha_j \geqslant 0, \quad j = k+1, \ldots, e, \\
& s_{i0}^- \geqslant 0, \quad i = 1, \ldots, m, \\
& s_{r0}^+ \geqslant 0, \quad r = 1, \ldots, s.
\end{aligned} \tag{A.2}$$

Since the optimal value of (4) is e–k, any feasible solution of (A.1) satisfies $-\sum_{i=1}^{m} v_i x_{ij} + \sum_{r=1}^{s} \mu_r y_{rj} < 0$, $j = k+1, \ldots, e$, (otherwise, the optimal value of (4) would be lower than e–k, i.e., we could find a hyperplane supported by more than k extreme efficient units). Consequently, at optimum we will have that $\alpha_{k+1} = \cdots = \alpha_e = 0$, by the complementary slackness theorem. Therefore, to obtain the optimal value of (A.2) we can equivalently solve the following LP problem:

$$\begin{aligned}
\min \quad & \theta_0 \\
\text{s.t.} \quad & \sum_{j=1}^{k} \lambda_j x_{ij} \leqslant \theta_0 x_{i0} - s_{i0}^-, \quad i = 1, \ldots, m, \\
& \sum_{j=1}^{k} \lambda_j y_{rj} \geqslant s_{r0}^+, \quad r = 1, \ldots, s,
\end{aligned}$$

$$\sum_{i=1}^{m} s_{i0}^{-} + \sum_{r=1}^{s} s_{r0}^{+} \geqslant 1,$$
$$\theta_0 \quad \text{free},$$
$$\lambda_j \quad \text{free}, \quad j = 1, \ldots, k,$$
$$s_{i0}^{-} \geqslant 0, \quad i = 1, \ldots, m,$$
$$s_{r0}^{+} \geqslant 0, \quad r = 1, \ldots, s. \tag{A.3}$$

By contradiction, suppose that the optimal value z_0^* of (A.1) is not strictly positive. Taking into account the strong duality conditions, $z_0^* = 0$ implies that the optimal value of (A.2) is zero, which means that $\theta_0^* = 0$ in (A.3). Hence, there exist $\lambda_j^* \in R$, $j = 1, \ldots, k$, $s_{i0}^{-*} \geqslant 0$, $i = 1, \ldots, m$, and $s_{r0}^{+*} \geqslant 0$, $r = 1, \ldots, s$, such that

$$\sum_{j=1}^{k} \lambda_j^* x_{ij} \leqslant -s_{i0}^{-*}, \quad i = 1, \ldots, m,$$
$$\sum_{j=1}^{k} \lambda_j^* y_{rj} \geqslant s_{r0}^{+*}, \quad r = 1, \ldots, s,$$
$$\sum_{i=1}^{m} s_{i0}^{-*} + \sum_{r=1}^{s} s_{r0}^{+*} \geqslant 1.$$

The last inequality allows us to state that there exists either $l \in \{1, \ldots, m\}$ such that $s_{l0}^{-*} > 0$ or $p \in \{1, \ldots, s\}$ such that $s_{p0}^{+*} > 0$. This, together with the remaining inequalities, leads us to assert that there exist j_0, $j_1 \in \{1, \ldots, k\}$, $j_0 \neq j_1$, such that $\lambda_{j_0}^* > 0$ and $\lambda_{j_1}^* < 0$. To show this, let us assume, for example, that $s_{p0}^{+*} > 0$. Then $\sum_{j=1}^{k} \lambda_j^* y_{pj} \geqslant s_{p0}^{+*} > 0$ and there will exist $j_0 \in \{1, \ldots, k\}$ such that $\lambda_{j_0}^* > 0$. From both this one and the first group of inequalities above, it follows that for each $i \in \{1, \ldots, m\}$ it holds $0 < \lambda_{j_0}^* x_{ij_0} \leqslant -\sum_{\substack{j=1 \\ j \neq j_0}}^{k} \lambda_j^* x_{ij}$, and so we have that there exists $j_1 \in \{1, \ldots, k\}$, $j_1 \neq j_0$, satisfying $\lambda_{j_1}^* < 0$. A similar reasoning can be followed if $s_{l0}^{-*} > 0$ is assumed.

Therefore, it can be concluded that:

$$\sum_{\lambda_j^* > 0} \lambda_j^* x_{ij} \leqslant \sum_{\lambda_j^* < 0} |\lambda_j^*| x_{ij} - s_{i0}^{-*}, \quad i = 1, \ldots, m,$$
$$\sum_{\lambda_j^* > 0} \lambda_j^* y_{rj} \geqslant \sum_{\lambda_j^* < 0} |\lambda_j^*| y_{rj} + s_{r0}^{+*}, \quad r = 1, \ldots, s$$

with each of these four sums being strictly positive.

Consequently, if we define the following two points $(\tilde{x}_1, \ldots, \tilde{x}_m, \tilde{y}_1, \ldots, \tilde{y}_s)$ and $(\bar{x}_1, \ldots, \bar{x}_m, \bar{y}_1, \ldots, \bar{y}_s)$ where:

$$\tilde{x}_i = \sum_{\lambda_j^* < 0} |\lambda_j^*| x_{ij}, \;\; i = 1, \ldots, m, \quad \text{and} \quad \tilde{y}_r = \sum_{\lambda_j^* < 0} |\lambda_j^*| y_{rj}, \;\; r = 1, \ldots, s,$$

and

$$\bar{x}_i = \sum_{\lambda_j^* > 0} |\lambda_j^*| x_{ij}, \;\; i = 1, \ldots, m, \quad \text{and} \quad \bar{y}_r = \sum_{\lambda_j^* > 0} |\lambda_j^*| y_{rj}, \;\; r = 1, \ldots, s,$$

which are two efficient ones, since they are on the facet spanned by $\{\mathrm{DMU}_1, \ldots, \mathrm{DMU}_k\}$, we will have two efficient points such that one of them is dominated by the other. This is clearly a contradiction. To avoid this contradiction we must have $z_0^* > 0$ for any extreme efficient unit.

To conclude the second part of the proof, if DMU_0 is not assumed to be an extreme efficient unit, then (A.1) will have the following additional equality constraint $-\sum_{i=1}^{m} v_i x_{i0} + \sum_{r=1}^{s} \mu_r y_{r0} = 0$, which is associated with this unit. See point C in Fig. 1. Consequently, the index j in (A.3) will vary over $\{0, 1, \ldots, k\}$. Besides, there will exist some scalars $a_j \geqslant 0$, $j = 1, \ldots, k$, at least one of them being strictly positive, such that

$$x_{i0} = \sum_{j=1}^{k} a_j x_{ij}, \quad i = 1, \ldots, m, \quad \text{and} \quad y_{r0} = \sum_{j=1}^{k} a_j y_{rj}, \quad r = 1, \ldots, s.$$

Now, again assuming $\theta_0^* = 0$ in (A.3), substituting the expressions above into the constraints of this problem, and doing some simple operations we will be in the same previously analyzed situation that forced z_0^* to be strictly positive. □

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