Control Systems Engineering

(Chapter 3. Modeling in the Time Domain)

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Introduction

- In this lesson, you will learn the following :
  - How to find a mathematical model, called a state-space representation, for linear time invariant system
  - How to convert between transfer function and state-space models
  - How to linearize a state-space representation

Introduction

- Two approaches are available for the analysis and design of feedback control systems
  - The first is known as the classical or frequency-domain technique
    - This approach is based on converting a system's differential equation to a transfer function
    - The primary disadvantage of the classical approach is its limited applicability
    - It can be applied only linear time-invariant systems
    - But this approach rapidly provides stability and transient response information

Introduction

- Next, the state-space approach (also referred to as the modern or time-domain approach) is a unified method for modeling, analyzing and designing a wide range of systems
  - We can use the state-space approach both linear and nonlinear systems
  - Also it can handle the systems with nonzero initial conditions

State-space Representation

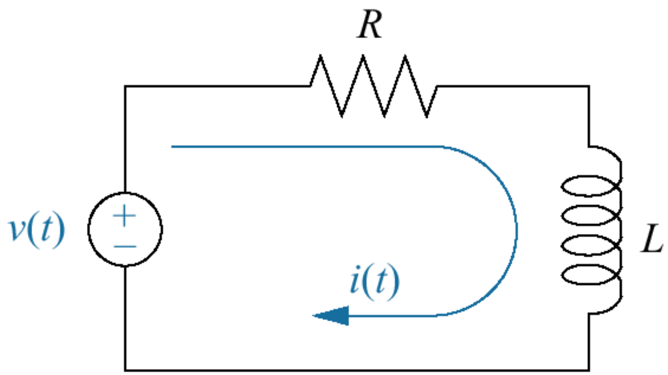
- Select a particular subset of all possible system variables
  - Call the variables in this subset as state variables
- For an n^{th}-order system, write n simultaneous, 1st-order differential equations in terms of the state variables
  - Call this system of simultaneous differential equations as state equations

State-space Representation

- Algebraically combine the state variables with the system's input and find all other system variables for $t \geq t_0$
  - Call this algebraic equation as the output equation
- Consider the state equations and the output equations as a viable representation of the system
  - Call this representation of the system as a state-space representation(state equation + output equation)

RL network

- Let us now follow the steps for state-space representation through an example
  - Consider RL network shown in figure with an initial current of $i(0)$



    - Select the current as state variable
    - Write the loop equation

$$L\frac{di(t)}{dt} + Ri(t) = v(t)$$

RL network

- Take the Laplace transform with including the initial conditions

$$L[sI(s) - i(0)] + RI(s) = V(\mathrm{s})$$

- Assuming the input, $v(t)$, to be a unit step, $u(t)$, whose Laplace transform is $V(s)=1/s$, we solve for $I(s)$ and get

$$I(s) = \frac{1}{s(Ls+R)} + \frac{Li(0)}{Ls+R} = \frac{1}{L}\left\{\frac{A}{s} + \frac{B}{s+R/L}\right\} + \frac{Li(0)}{Ls+R}$$

$$= \frac{1}{R}\left(\frac{1}{s} - \frac{1}{s+R/L}\right) + \frac{i(0)}{s+R/L} \qquad \text{where } A = L/R,\ B = -L/R$$

$$\Rightarrow \quad i(t) = \frac{1}{R}\left[1 - e^{-(R/L)t}\right] + i(0)e^{-(R/L)t}$$

  - ✓ $i(t)$ is a subset of all possible network variables that we can find if we know its initial condition, $i(0)$, and the input $v(t)$

RL network

- ✓ Thus, $i(t)$ is a state variable, and the loop equation is a state equation
- Knowing the state variable, $i(t)$, and the input $v(t)$, we can find the value, or state, of any network variable at any time $t \geq t_0$
  - ✓ Thus, the algebraic equations of $v_R(t)$ and $v_L(t)$ are the output equations

$$v_R(t) = Ri(t), \quad v_L(t) = v(t) - Ri(t)$$

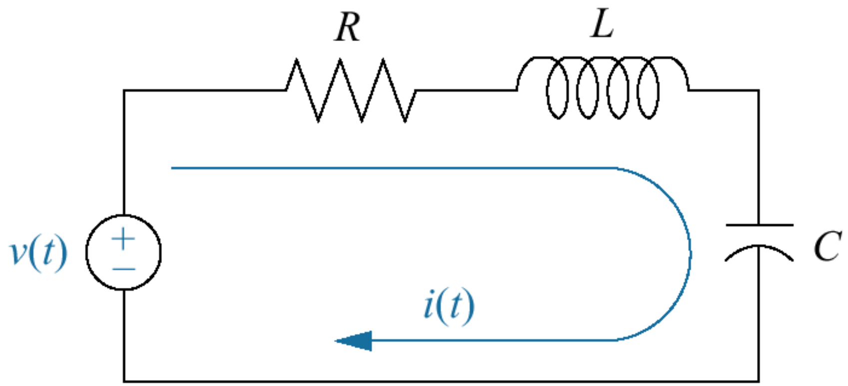
- Combining the state equation and the output equation is called the state-space representation

$$i(t) = \frac{v_R(t)}{R} = \frac{v(t) - v_L(t)}{R},$$

$$\frac{di(t)}{dt} = \frac{v(t) - Ri(t)}{L}$$

RLC network

- Let us now extend our observations to a 2nd-order system and find the state-space representation of this 2nd-order system



  - Since the network is 2nd-order, two simultaneous 1st-order differential equation are needed to solve for two state variables
  - Select $i(t)$ and $q(t)$(the charge on the capacitor) as the two state variables

RLC network

- Write the loop equation

$$L\frac{di(t)}{dt}+Ri(t)+\frac{1}{C}\int i(t)dt=v(t)$$

- Converting the equation in terms of $i(t)=\dfrac{dq(t)}{dt}$, we get

$$L\frac{dq^2(t)}{dt^2}+R\frac{dq(t)}{dt}+\frac{1}{C}q(t)=v(t)$$

- An n^{th}-order differential equation can be converted to n simultaneous 1^{st}-order differential equation of the form

$$\frac{dx_i(t)}{dt}=a_{i1}x_1(t)+a_{i2}x_2(t)+\cdots+a_{in1}x_n(t)+b_i f(t)$$

, which is a linear combination of the state variables and the input, $f(t)$

RLC network

- Summarizing the two resulting equations, we get

$$\frac{dq(t)}{dt} = i(t),$$

$$\frac{di(t)}{dt} = \frac{dq^2(t)}{dt^2} = -\frac{1}{LC}q(t) - \frac{R}{L}i(t) + \frac{1}{L}v(t)$$

  - These equations are the state equations

- From these two state variables, we can solve for all other network variables
  - For example, the voltage across the inductor can be written in terms of the solved state variables and the input as

$$V_L(t) = -\frac{1}{C}q(t) - Ri(t) + v(t)$$

    - ✓ This equation is an output equation

- The combined state equation and output equation is called as state-space representation

RLC network

- Is there any restriction on the choice of state variables? YES!
  - No state variable can be chosen if it can be expressed as a linear combination of the other state variables
    - For example, if $V_R(t)$ is chosen as a state variable, then $i(t)$ can not be chosen, because $V_R(t)$ can be written as a linear combination of $i(t)$, namely

$$V_R(t) = Ri(t)$$

    - Under these circumstances we say that the state variables are linearly independent
    - State variables must be linearly independent; that is, no state variable can be written as a linear combination of all the other state variables

RLC network

- The state and output equations can be written in vector-matrix form if the system is linear
  - Thus, the state-space representation of the RLC network given can be written as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \quad y = \mathbf{C}\mathbf{x} + Du$$

where

$$\dot{\mathbf{x}} = \begin{bmatrix} dq/dt \\ di/dt \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} q \\ i \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix},$$

(time derivative of the state vector) (state vector) (system matrix) (input matrix)

$$y = V_L(t), \quad \mathbf{C} = \begin{bmatrix} -\frac{1}{C} & -R \end{bmatrix}, \quad D = 1, \quad u = v(t)$$

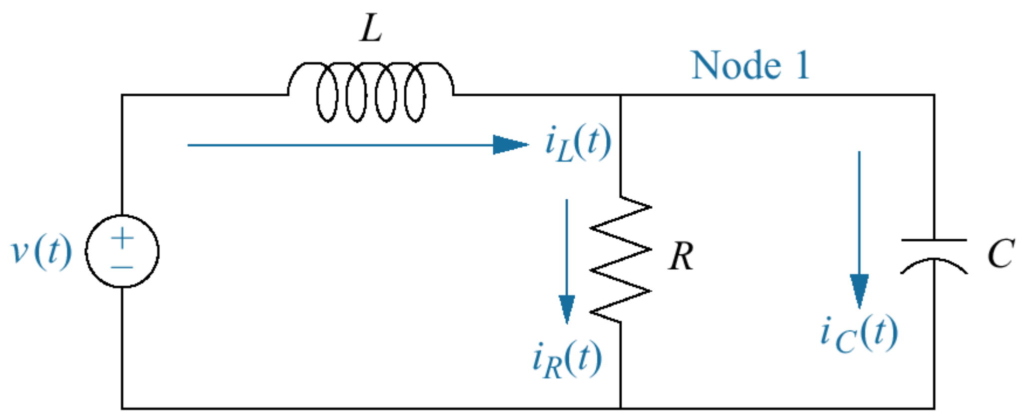
(output vector) (output matrix) (feedforward matrix)

RLC network

- The first step in representing a system is to select the state vector, which must be chosen according the following considerations:
  - A minimum number of state variables must be selected as components of the state vector
  - The components of the state vector (that is, this minimum number of state variables) must be linearly independent
  - How do we know the minimum number of state variables to select?
    - Typically, the minimum number required equals to the order of differential equation describing the system

Applying the State-Space Representation

- Example:
  - Given the electrical network of figure below, find a state-space representation if the output is the current through the resistor



  - Solution:
    - Step 1:
      - ✓ Label all of the branch currents in the network.
      - ✓ These include $i_L(t), i_R(t)$ and $i_C(t)$ as shown in the figure

Applying the State-Space Representation

- Step 2:
  - ✓ Select the state variables by writing the derivative equation for all energy storage elements, that is, the inductor and capacitor
  - ✓ Thus, $$C\frac{dv_C(t)}{dt} = i_C(t), \quad L\frac{di_L(t)}{dt} = v_L(t)$$
  - ✓ Using these two equations, choose the state variables as the quantities that are differentiated, namely $v_C(t)$ and $i_L(t)$
- Step 3:
  - ✓ Apply the network theory to obtain $v_L(t)$ and $i_C(t)$ in terms of the state variables
  - ✓ At Node 1: $$i_C(t) = -i_R(t) + i_L(t) = -\frac{v_C(t)}{R} + i_L(t)$$
  - ✓ Around the outer loop: $v_L(t) = -v_C(t) + v(t)$
- Step 4:
  - ✓ Using the equations we wrote in the previous steps, obtain the following state equations:

Applying the State-Space Representation

$$C\frac{dv_C}{dt} = -\frac{1}{R}v_C + i_L, \quad L\frac{di_L}{dt} = -v_C + v(t)$$

$$\Rightarrow \frac{dv_C}{dt} = -\frac{1}{RC}v_C + \frac{1}{C}i_L, \quad \frac{di_L}{dt} = -\frac{1}{L}v_C + \frac{1}{L}v(t)$$

- Step 5:
  - ✓ Find the output equation. Since the output is $i_R(t)$, $i_R(t) = \frac{1}{R}v_c(t)$
  - ✓ The final result for the state–space representation in vector-matrix form is

$$\begin{bmatrix} \dot{v}_C \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} v(t), \quad i_R = \begin{bmatrix} \frac{1}{R} & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix}$$

Converting a Transfer Function to State Space

- We will learn how to convert a transfer function representation to a state-space representation
  - Let us begin by showing how to represent a general n^{th}-order linear differential equation with constant coefficients in state-space in the phase variable-form
    - Phase variable: A set of state variable where each subsequent state variable is defined to be the derivative of the previous state variable
  - We will then show how to apply this representation to transfer function
  - General differential equation:

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_0 u$$

Converting a Transfer Function to State Space

- A convenient way to choose state variables is to choose the output, $y(t)$, and its $(n\text{-}1)$ derivatives as the state variables
  - This choice is called phase-variables choice.
- Choosing the state variables, x_i, we get

$$x_1 = y, \quad x_2 = \frac{dy}{dt}, \quad x_3 = \frac{d^2y}{dt^2}, \quad \cdots, \quad x_n = \frac{d^{n-1}y}{dt^{n-1}}$$

- Differentiating both sides yields

$$\dot{x}_1 = \frac{dy}{dt}, \quad \dot{x}_2 = \frac{d^2y}{dt^2}, \quad \dot{x}_3 = \frac{d^3y}{dt^3}, \quad \cdots, \quad \dot{x}_n = \frac{d^ny}{dt^n}$$

- The state equations are evaluated as

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \cdots, \quad \dot{x}_{n-1} = x_n, \quad \dot{x}_n = -a_0x_1 - a_1x_2 \cdots - a_{n-1}x_n + b_0u$$

Converting a Transfer Function to State Space

- The state–space representation in vector-matrix form is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} u$$

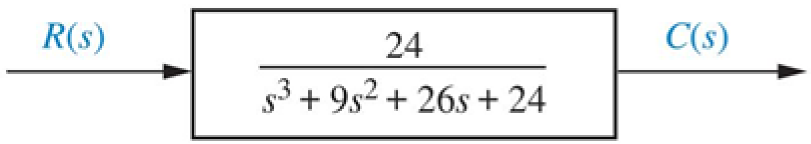
$$y = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

Converting a Transfer Function to State Space

- In summary, to convert a transfer function into state equations in phase-variable form, we first convert the transfer function to a differential equation by cross-multiplying and taking the inverse Laplace transform, assuming zero initial conditions
- Then, we represent the differential equation in state-space in phase-variable form
- An example illustrates the process

Example:

- Find the state-space representation in phase-variable form for the transfer function shown in the figure below



(a)

Converting a Transfer Function to State Space

- Solution:
  - Step 1:
    - Find the associated differential equation
    - Since $\frac{C(s)}{R(s)} = \frac{24}{s^3 + 9s^2 + 26s + 24}$, cross-multiplying yields

$$\left(s^3 + 9s^2 + 26s + 24\right)C(s) = 24R(s)$$

    - The corresponding differential equation is found by taking the inverse Laplace transform, assuming zero initial conditions:

$$\dddot{c} + 9\ddot{c} + 26\dot{c} + 24c = 24r$$

  - Step 2:
    - Select the state variables
    - Choosing the state variables as successive derivatives, we get

$$\text{(State variables)}\begin{cases} x_1 = c \\ x_2 = \dot{c} \\ x_3 = \ddot{c} \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + 24r \end{cases} \text{(System equations)}$$

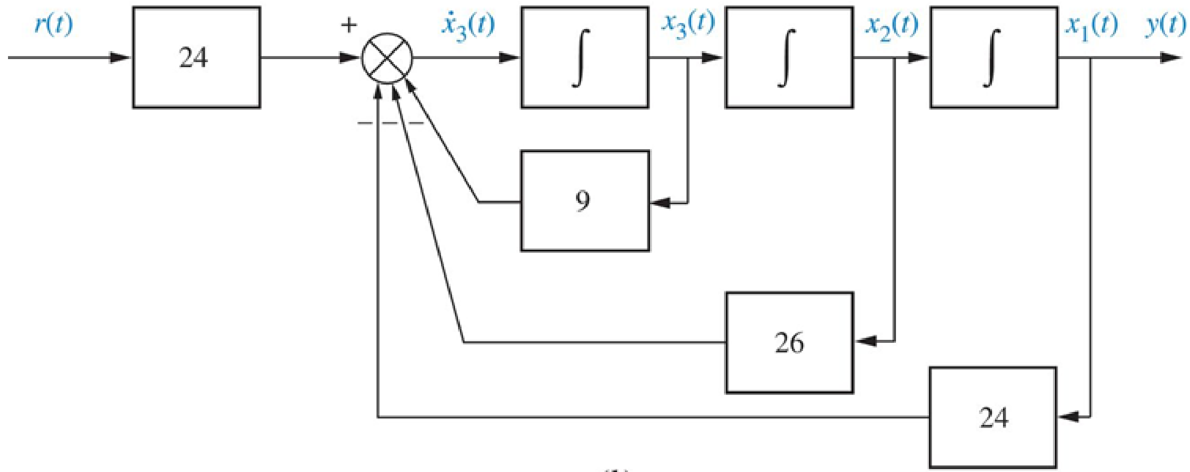
$$y = c = x_1 \quad \text{(Output equation)}$$

Converting a Transfer Function to State Space

- ✓ The state–space representation in vector matrix form is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r, \quad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

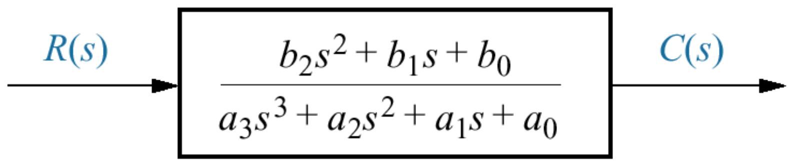
- ✓ In this point we can create an equivalent block diagram of the system to visualize the state variables
- ✓ We draw three integral blocks as shown in figure below and label each output as one of the state variables



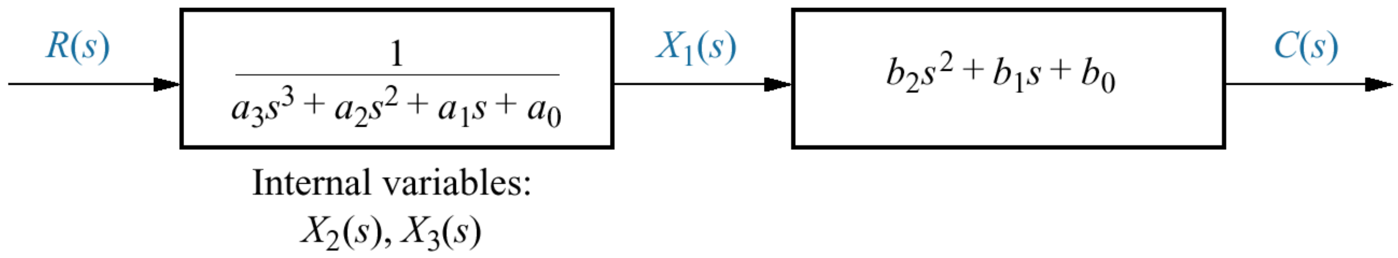
(b)

Converting a Transfer Function to State Space

- Converting a Transfer Function with polynomial in numerator
  - The numerator and denominator can be handled separately



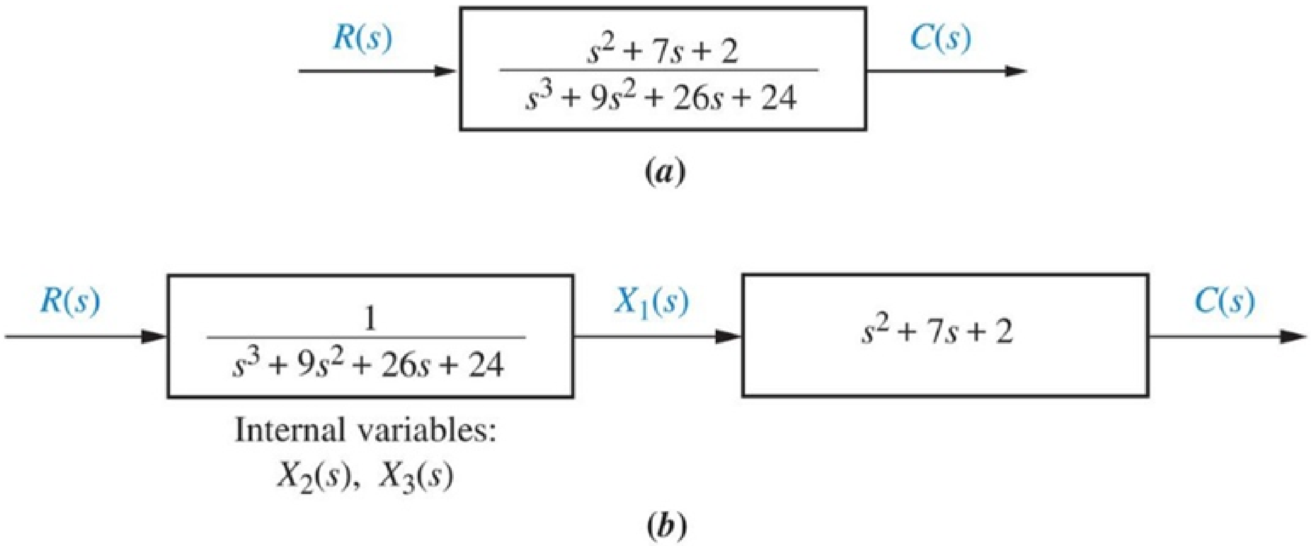
(a)



(b)

Converting a Transfer Function to State Space

- Example: Find the state-space representation of the transfer function shown in Figure



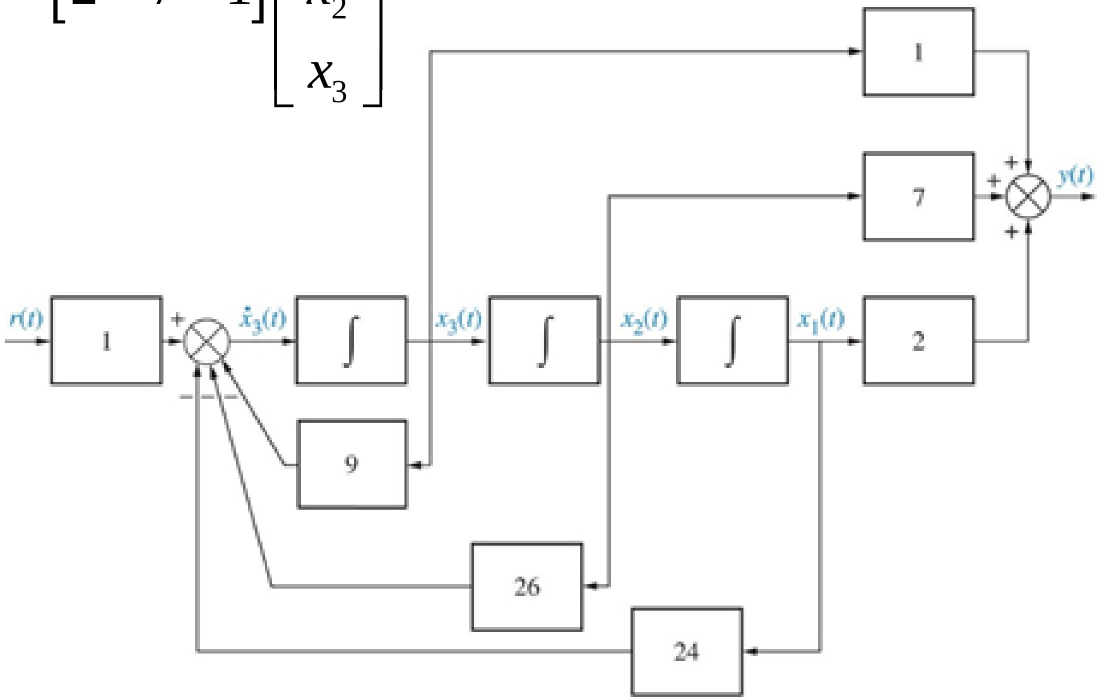
- (Solution)
  - ✓ The state–space representation in vector matrix form is

Converting a Transfer Function to State Space

$$C(s) = (s^2 + 7s + 2)X_1(s) \Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = x_3 + 7x_2 + 2x_1 \Rightarrow y = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(Equivalent block diagram)



Converting a Transfer Function to State Space

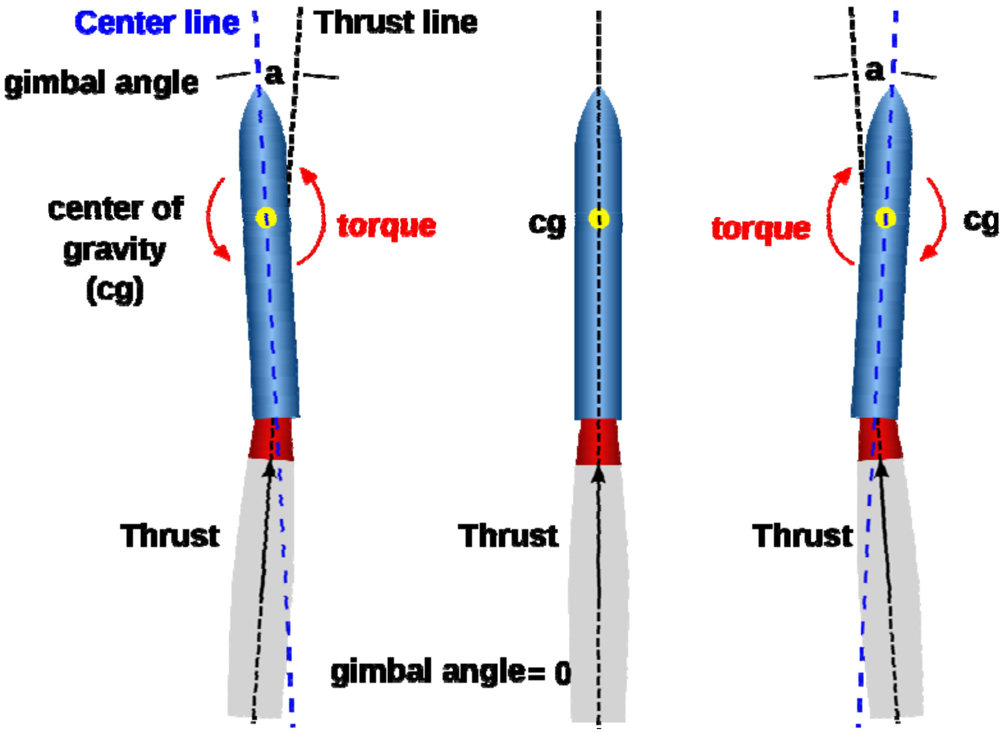
Example:

- A missile in flight, as shown in Figure P3.11, is subject to several forces: thrust, lift, drag, and gravity
- The missile flies at an angle of attack, α , from its longitudinal axis, creating lift
- For steering, the body angle from vertical, ϕ , is controlled by rotating the engine at the tail
- The transfer function relating the body angle, ϕ, to the angular displacement, δ, of the engine is of the form

$$\frac{\Phi(s)}{\delta(s)} = \frac{K_a s + K_b}{K_3 s^3 + K_2 s^2 + K_1 s + K_0}$$

- Represent the missile steering control in state space

Converting a Transfer Function to State Space



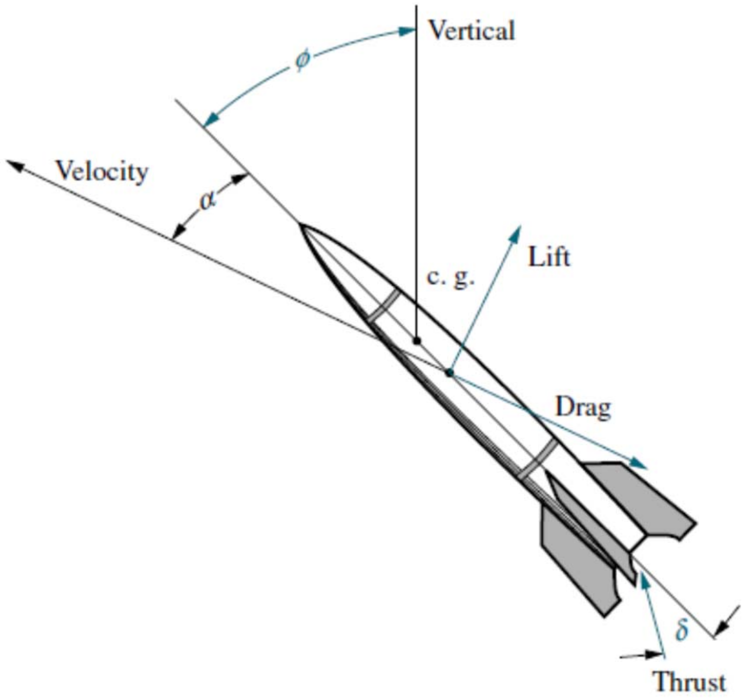
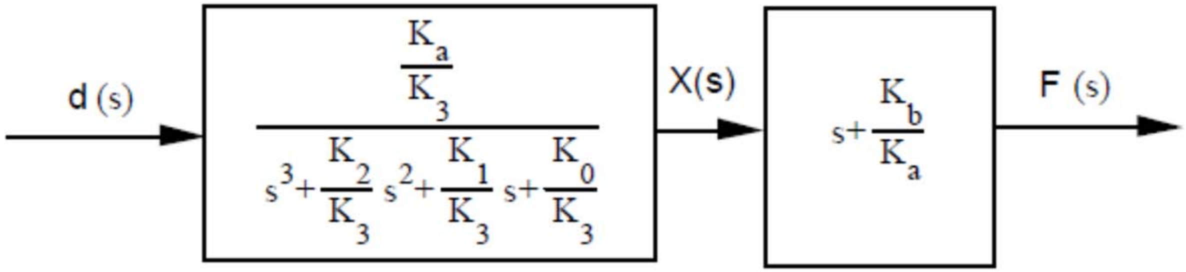


FIGURE P3.11 Missile

- Thrust is the force which moves an aircraft through the air
- Thrust is used to overcome the drag of an airplane, and to overcome the weight of a rocket
- Thrust is generated by the engines of the aircraft

Converting a Transfer Function to State Space

- Solution:
  - The equivalent cascade transfer function is as shown below



  - For the first box, $\dddot{x}+\dfrac{K_2}{K_3}\ddot{x}+\dfrac{K_1}{K_3}\dot{x}+\dfrac{K_0}{K_3}x=\dfrac{K_a}{K_3}\delta(t)$
  - Selecting the phase variables as the state variables: $\begin{cases} x_1 = x \\ x_2 = \dot{x} \\ x_3 = \ddot{x} \end{cases}$
  - Writing the state and output equations:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -\frac{K_0}{K_3}x_1 - \frac{K_1}{K_3}x_2 - \frac{K_2}{K_3}x_3 + \frac{K_a}{K_3}\delta(t)$$

Converting a Transfer Function to State Space

$$y = \phi(t) = \dot{x} + \frac{K_b}{K_a} x = \frac{K_b}{K_a} x_1 + x_2$$

- In vector-matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{K_0}{K_3} & -\frac{K_1}{K_3} & -\frac{K_2}{K_3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{K_a}{K_3} 4 \end{bmatrix} \delta(\mathrm{t}),$$

$$y = \begin{bmatrix} \frac{K_b}{K_a} & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Converting from state space to a Transfer Function

- Given the state and output equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \quad y = \mathbf{C}\mathbf{x} + Du$$

  - take the Laplace transform assuming zero initial conditions:

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s), \quad \mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)$$

  - Solving for $\mathbf{X}(s)$ yields

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}\mathbf{U}(s) \quad \Rightarrow \quad \mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

    - where $\mathbf{I}$ is identity matrix

  - Substituting the equation into equation $\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)$ yields

$$\mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\mathbf{U}(s)$$

$$\mathbf{T}(s) = \frac{\mathbf{Y}(s)}{\mathbf{U}(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

Converting from state space to a Transfer Function

Example:

- Given the system defined by the following equations, find the transfer function $\mathbf{T}(s) = \mathbf{Y}(s)/\mathbf{U}(s)$, where $\mathbf{U}(s)$ is the input and $\mathbf{Y}(s)$ is the output

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- Solution:
  - First find $(s\mathbf{I} - \mathbf{A})$

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}$$

Converting from state space to a Transfer Function

- Now form $(s\mathbf{I} - \mathbf{A})^{-1}$

$$(s\mathbf{I}-\mathbf{A})^{-1} = \frac{\begin{bmatrix} s^2+3s+2 & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -2(s+1) & s^2 \end{bmatrix}}{s^3+3s^2+2s+1}$$

- Substituting $(s\mathbf{I} - \mathbf{A})^{-1}, \mathbf{B}, \mathbf{C}$ and D into equation, we obtain the final result transfer function

$$\mathbf{B} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, D = 0$$

Converting from state space to a Transfer Function

$$\mathbf{T}(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} + D$$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \frac{\begin{bmatrix} s^2+3s+2 & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -2(s+1) & s^2 \end{bmatrix}}{s^3+3s^2+2s+1} \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{s^3+3s^2+2s+1} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 10(s^2+3s+2) \\ -10 \\ -10s \end{bmatrix} = \frac{10(s^2+3s+2)}{s^3+3s^2+2s+1}$$

Laplace Transform Solution of State Equation

- Math reference for Inverse Matrix:
  - Let $A = \left[a_{ij} \right]$ be an $n \times n$ square matrix
  - Define an $n \times n$ matrix $B = \left[b_{ij} \right]$ by setting

$$b_{ij} = \frac{1}{|A|} (-1)^{i+j} M_{ji}$$

    - where M_{ji} is the minor formed from A by deleting row j and column i of A
  - Then, $B = A^{-1}$

Laplace Transform Solution of State Equation

- Given the state and output equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \quad y = \mathbf{C}\mathbf{x} + Du$$

  - Taking the Laplace transform of both sides of the state equations yields

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

  - In order to separate $\mathbf{X}(s)$, replace $s\mathbf{X}(s)$ with $s\mathbf{I}\mathbf{X}(s)$, where $\mathbf{I}$ is an $n \times n$ identity matrix, and n is the order of the system
  - Combining all of the $\mathbf{X}(s)$ terms, we get

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)$$

Laplace Transform Solution of State Equation

- Solving for $\mathbf{X}(s)$ by premultiplying both sides of the last equation by $(s\mathbf{I}-\mathbf{A})^{-1}$ yields

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}[\mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)]$$

$$= \frac{\mathrm{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})}[\mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)]$$

- Taking the Laplace transform of the output equation yields

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)$$

Eigenvalues and Transfer Function Poles

- Example:
  - Given the system represented in state space by equations

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-t} \qquad y = [\,1 \quad 1 \quad 0\,]\,\mathbf{x} \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

  - do the following:
    - ✓ a. Solve the preceding state equation and obtain the output for the given input
    - ✓ b. Find the eigenvalues and the system poles
  - Solution:

a) Remember the equation $X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)$

$$(sI - A) = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 24 & 26 & s+9 \end{bmatrix} \Rightarrow (sI - A)^{-1} = \frac{\begin{bmatrix} (s^2+9s+26) & s+9 & 1 \\ -24 & s^2+9s & s \\ -24s & -26s+24 & s^2 \end{bmatrix}}{s^3 + 9s^2 + 26s + 24}, \quad \mathrm{U}(\mathrm{s}) = \frac{1}{\mathrm{s}}$$

Eigenvalues and Transfer Function Poles

we get $X_1 = \dfrac{(s^3 + 10s^2 + 37s + 29)}{(s+1)(s+2)(s+3)(s+4)} \xleftrightarrow{L^{-1}} x_1(t)$

$$X_2 = \frac{(2s^2 - 21s - 24)}{(s+1)(s+2)(s+3)(s+4)} \qquad X_3 = \frac{s(2s^2 - 21s - 24)}{(s+1)(s+2)(s+3)(s+4)}$$

$$Y(s) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{bmatrix} = X_1(s) + X_2(s)$$

$$\Rightarrow Y(s) = \frac{(s^3 + 12s^2 + 16s + 5)}{(s+1)(s+2)(s+3)(s+4)} = \frac{-6.5}{s+2} + \frac{19}{s+3} - \frac{11.5}{s+4}$$

where a pole at -1 canceled a zero at -1

Taking the inverse Laplace transform :

$$y(t) = -6.5e^{-2t} + 19e^{-3t} - 11.5e^{-4t}$$

b) The roots of det(sI-A)=0 give us both the poles of the system and the eigenvalues which are -2, -3 and -4.

Homework Assignment #3

1. Represent the electrical network shown in Figure P3.1 in state space, where $v_o(t)$ is the output.

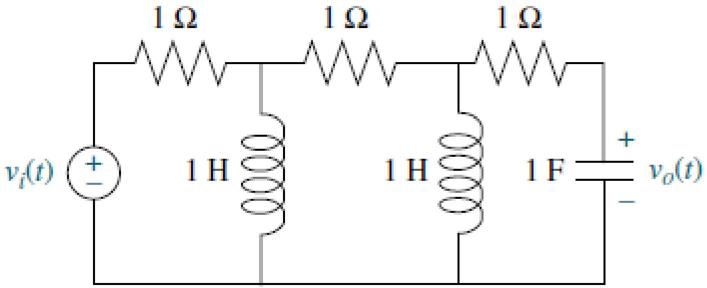


FIGURE P3.1

2. Represent the electrical network shown in Figure P3.2 in state space, where $i_R(t)$ is the output.

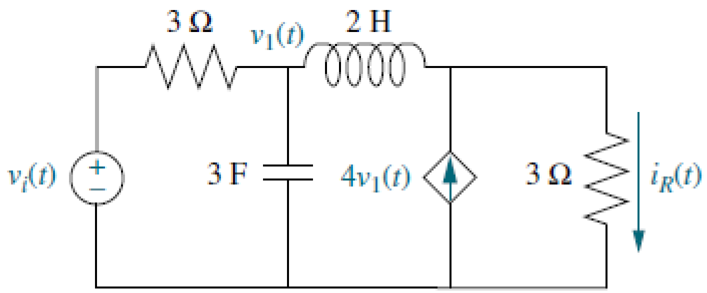


FIGURE P3.2

Homework Assignment #3

9. Find the state-space representation in phase-variable form for each of the systems shown in Figure P3.8.

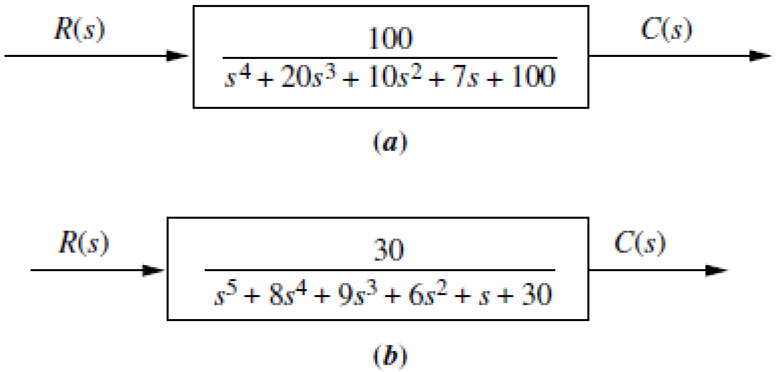


FIGURE P3.8

11. For each system shown in Figure P3.9, write the state equations and the output equation for the phase-variable representation.

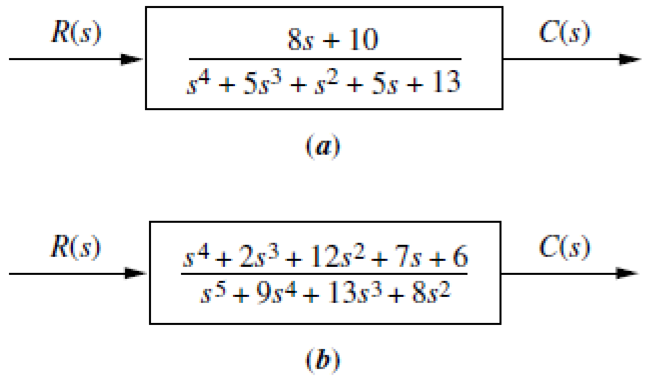


FIGURE P3.9

Homework Assignment #3

14. Find the transfer function $G(s) = Y(s)/R(s)$ for each of the following systems represented in state space:

a. $$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -2 & -5 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} r$$

$$y = [1 \quad 0 \quad 0]\mathbf{x}$$

b. $$\dot{\mathbf{x}} = \begin{bmatrix} 2 & -3 & -8 \\ 0 & 5 & 3 \\ -3 & -5 & -4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} r$$

$$y = [1 \quad 3 \quad 6]\mathbf{x}$$

c. $$\dot{\mathbf{x}} = \begin{bmatrix} 3 & -5 & 2 \\ 1 & -8 & 7 \\ -3 & -6 & 2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} r$$

$$y = [1 \quad -4 \quad 3]\mathbf{x}$$

39. Solve the following state equation and output equation for $y(t)$, where $u(t)$ is the unit step. Use the Laplace transform method.

$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & 0 \\ -1 & -1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$y = [0 \quad 1]\mathbf{x};\ \mathbf{x}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

40. Solve for $y(t)$ for the following system represented in state space, where $u(t)$ is the unit step. Use the Laplace transform approach to solve the state equation.

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -6 & 1 \\ 0 & 0 & -5 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(t)$$

$$y = [0 \ 1 \ 1]\mathbf{x};\ \mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$